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Proof Theory for Linear Lattices

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We make a study of lattices representable by commuting equivalence relations, which we call *linear* lattices. We develop a proof theory for implications valid in linear lattices, which differs from classical proof theories in that its deductions transform representative graphs rather than well-formed formulas. Using graph-theoretic arguments, we establish a duality theorem and a normal form theorem for this proof theory. © 1985 Academic Press, Inc.

0.0. INTRODUCTION

The classical theory of lattices, as it evolved out of the nineteenth century through the work of Boole, Charles Saunders Peirce, and Schröder, and later in the work of Dedekind, Ore, Birkhoff, Von Neumann, Dilworth, and others, can today be viewed as essentially the study of two classes of lattices, together with their variants and their implications for their naturally occurring models. These are the classes of distributive lattices, whose natural models, which they capture exactly, are systems of sets or, from another point of view, of logical propositions; and modular lattices, whose natural but by no means only models are quotient structures of algebraic entities such as groups, rings, modules, and vector spaces.

In actuality, the lattices of normal subgroups of a group, ideals of a ring, or subspaces of a vector space are more than modular; as Birkhoff and Dubreil–Jacotin were first to observe, they are lattices of equivalence relations which commute relative to the operation of composition of relations. The combinatorial properties of lattices of commuting equivalence relations are not mere consequences of their modularity, but rather the opposite; the consequences of the modular law derived since Dedekind, who originally formulated it, have mainly been guessed on the basis of examples which were lattices of commuting equivalence relations.

This paper begins a study of lattices of commuting equivalence relations, which we have named *linear* lattices, a term suggested by G.-C. Rota for its evocation of the archetypal example of projective geometry. It is predicated on the supposition that in the linear lattice case, there is hope of carrying

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out the dream of Birkhoff and Von Neumann, to understand modular lattices through a "modular" extension of classical logic, just as distributive lattices had been so effectively understood through the constellation of ideas connecting classical propositional logic, the theory of sets, and visualization via the device of Venn diagrams. Accordingly, our main concern has been to develop the proof theory for a logic of linear lattices, along lines that we believe to be new.

The fundamental need for a spatial visualization of statements pertaining to a logic was recognized by Birkhoff and Von Neumann when they proposed replacing the visual aid of Venn diagrams with the visual aid of configurations of subspaces in a vector space. Unfortunately, no one has ever been able to visualize any but the simplest propositions on subspaces of a vector space, and the suggestion met with limited success.

Our proposal is to visualize statements pertaining to linear lattices with the aid of series–parallel networks (extensively studied in combinatorics and circuit theory). On this basis we proceed to develop a full-fledged proof theory for the logic of linear lattices.

It is by now understood and accepted that the central fact of classical logic is a decision procedure (due to Herbrand, Gentzen, Beth, Hintikka, Smullyan, and others, and for which there is unfortunately no generic name) whereby an algorithm is given for the verification of logical statements which either ends with the actual verification, or else goes on indefinitely, but produces a counterexample in the process. The actual decidability of logical statements is a secondary issue to the fact that such an algorithm either verifies the given statement or refutes it, in a manner fully consonant with the expectations of logical reasoning.

Briefly, our procedure is to represent a lattice implication whose truth is to be verified for linear lattices by pairs of series–parallel networks corresponding to the hypotheses and conclusion of the implication. The truth of the conclusion, subject to the hypotheses, is then seen to follow from the existence of a generalized graph homomorphism mapping one of the series–parallel networks to the other. We describe a construction which either produces such a homomorphism at some finite stage, or else continues indefinitely and in so doing, automatically produces a counterexample. This construction can be viewed as an analog of Herbrand's theorem, or of Gentzen's proof theory, for linear lattices.

In the second half of this paper, guided by the analogy of our result with Gentzenian proof theory, we derive a normal form theorem for proofs of inequalities in linear lattices, which may be viewed as a linear lattice analog of Gentzen's Hauptsatz for classical first-order logic. This is the deepest result of the present investigation, for while it falls short of establishing the decidability of the equational theory of linear lattices, it allows us to clearly enunciate what step needs to be taken to settle the question.

We stress the value of these results as a practical method of guessing, verifying, and visualizing linear lattice identities. A major application of our method is to finding and proving theorems of projective geometry relating to incidence of subspaces, independent of dimension. As illustrations of our methods, we produce a hierarchy of generalizations of Desargues' theorem, including a particularly elegant form of Jónsson's Arguesian identity. In addition, we single out a lattice inequality which we expect may settle Jónsson's question of whether all Arguesian lattices are linear.

We wish to emphasize also the relevance of the present work to the invariant theory of linear varieties, approached along the lines initiated by Gel'fand and Ponomarev in their influential papers on representations of free modular lattices and recently further developed by Herrmann, Huhn, Wille, and others. We expect that the remarkable structural features found by Gel'fand and Ponomarev in their linear (in the sense of linear algebra) quotients of free modular lattices will manifest themselves already in free linear lattices, in our sense of representability by commuting equivalence relations. If so, the proof-theoretic tools we have developed for linear lattices may contribute substantial insights and simplifications to this line of work.

This paper may be read as an argument for the contention that much of the combinatorial subtlety of synthetic projective geometry (typically, the Von Staudt/Von Neumann coordinatization theorem) resides in the combinatorics of commuting equivalence relations; and further that commutativity of equivalence relations can be understood by a parallel reasoning to the classical logical ideas that explain distributivity. It is our belief that the theory of linear lattices, because of its combinatorial elegance, its intuitively appealing proof theory, and its broad range of potential applications, may finally come to exert on algebra, combinatorics, and geometry the unifying influence that modular lattices, despite their great historical significance, failed to achieve.

Preliminaries

This section consists of basic terminology, definitions, and properties for lattices, linear lattices, graphs, and series–parallel graphs. The reader will find a quick reading now, with referal back later as necessary, adequate for most of this material.

One point deserving special attention is the definition of *series–parallel graph.* This concept is fundamental to the statement of each major result in this thesis. While it is a simple and natural concept, this fact may be more readily appreciated from examples than from the definition. Therefore, a reader not already familiar with series–parallel graphs would do well to look at Figs. 5, 6, 9 (first and last drawing), 10, and 11, as well as reading the text below. Attention should also be given to Lemma 1.1, which follows the description of series–parallel graphs. Lemma 1.1 contains the root connection between series–parallel graphs and linear lattices. The Main Theorem on Proof Theory of the next section relies heavily upon Lemma 1.1.

Lattices. Our terminology is mostly standard (see [1, 3, or 14]). By a *lattice polynomial* we mean a term in the similarity type $(\wedge, \vee)$ of arity (2,2) over a fixed alphabet $\mathscr{A} = \{a, b, ..., \}$. If P, Q are lattice polynomials, we have $P = Q$ iff $P \vee Q \leqslant P \wedge Q$ and $P \leqslant Q$ iff $P \vee Q = Q$ in any lattice, so we freely use the word *identity* to refer to an inequality $P \leqslant Q$.

We have occasion to substitute least and greatest element symbols 0 and 1 for some of the variables in an identity $P \leqslant Q$. Since 0 and 1 are not in our similarity type, what we shall mean is to substitute and then simplify by the rules $0 \wedge x = x \wedge 0 = 0$, $0 \vee x = x \vee 0 = x$, $1 \wedge x = x \wedge 1 = x$, $1 \vee x = x \vee 1 = 1$. If neither P nor Q simplifies to 0 or 1, we get a new identity which we consider the result of the substitution. Otherwise the result is one of two extra "identities" denoted T and F: T if P simplifies to 0 or Q simplifies to 1; F in the remaining cases. T is defined to hold in every lattice; F is defined to hold only in the one-element lattice.

If L is a lattice, L^d denotes the *dual lattice* with the same underlying set as L and the meet and join operations interchanged. If P is a lattice polynomial, P^d denotes the *dual polynomial* obtained from P by interchanging the operation symbols $\wedge$ and $\vee$. If $P \leqslant Q$ is an identity, the *dual identity* is $Q^d \leqslant P^d$.

Linear lattices. Let $p(S)$ denote the lattice of equivalence relations (or partitions) on the set S. A function $\rho: L \to p(S)$ is a representation of the lattice L by commuting equivalence relations if for all $x, y \in L$, $\rho(x \wedge y) = \rho(x) \cap \rho(y)$ and $\rho(x \vee y) = \rho(x) \circ \rho(y) = \rho(y) \circ \rho(x) = \rho(x) \vee \rho(y)$ (recall that for equivalence relations r, s: $r \circ s = r \vee s$ iff $r \circ s$ is an equivalence relation). Jónsson [36] called such a representation "type I." We prefer the more suggestive term *linear* representation, and call L a *linear lattice* if L has a faithful, i.e., injective, linear representation.

If $r \subseteq S \times S$ is any relation, we write $u \sim v[r]$ for $(u, v) \in r$. As above let $\rho: L \to p(S)$ be a linear representation. When ρ is understood from context and $x \in L$, we write $u \sim v[x]$ for $u \sim v[\rho(x)]$. Given an interpretation $j: \mathscr{A} \to L$ of our alphabet into L, we extend j by evaluation to an interpretation of all lattice polynomials into L and write $u \sim v[j(P)]$ for $u \sim v[\rho(P|_{a \leftarrow j(a)(a \in \mathscr{A})})]$.

If $\rho: L \to p(S)$ and $\rho': L' \to p(S')$ are linear representations with $S \cap S' = \varnothing$, their *sum* $\rho \oplus \rho': L \times L' \to p(S \cup S')$ defined by $\rho \oplus \rho'((x, x')) = \rho(x) \cup \rho(x')$ is also a linear representation. If $L = L'$ we also refer to $(\rho \oplus \rho') \circ \Delta: L \to S \cup S'$, where $\Delta: L \to L \times L$ is $\Delta(x) = (x, x)$, as the *sum* of the representations ρ, ρ' of L. Sums of arbitrary finite or infinite collections of representations are defined analogously.

Graphs. As usual, a *graph* G consists of a set $V(G)$ of *vertices*, a set $E(G)$ of *edges*, and an *incidence relation* $i(G) \subseteq E(G) \times V(G)$ such that each edge is incident with one or two vertices. Loops and multiple edges are allowed. Most graphs in this thesis have a *distinguished edge* ε_G (or just ε, when there is no danger of confusion) incident with *distinguished vertices* ε_{G0} and ε_{G1} (or just ε_0 and ε_1). We admit the possibility that $\varepsilon_0 = \varepsilon_1$. All graphs have *labelled edges*, i.e., come equipped with a label function $l: E(G)\backslash\varepsilon \to \mathscr{A}$.

We at times invoke certain common manipulations on graphs without explicitly defining them. Thus we say "identify vertices x, y of G" instead of "construct a graph G' with $V(G') = V(G)\backslash\{x, y\} \cup \{z\}$ where z is a new element, and $E(G') = E(G)$, defining $i(G')$ so that each edge incident in G with x or y is incident in G' with z and $i(G') \cap (E(G') \times (V(G)\backslash\{x, y\})) = i(G) \cap (E(G) \times (V(G)\backslash\{x, y\}))$." Other examples are "partition a vertex" (opposite of identifying vertices), "form a disjoint union of two graphs," and so on.

When there is an isomorphism between two graphs preserving the distinguished edge and vertices and the labelling, we usually regard them as identical.

A *subgraph* $H \subseteq G$ consists of a subset $E(H) \subseteq E(G)$ of the edges, together with the subset $V(H) \subseteq V(G)$ of those vertices incident with edges in $E(G)$. $i(H)$ is the restriction of $i(G)$ to $E(H) \times V(H)$. Note that subgraphs need not be induced. H inherits its labelling from G.

If $e \in E(G)$, the *deletion* $G\backslash e$ is the subgraph whose edge set is $E(G)\backslash e$ (note our convention is to write singleton sets without brackets). The *contraction* G/e is obtained by first identifying the vertices incident with e, if there are two of them, then deleting e. Deletions and contractions commute, so the notation $G/\{e, f, ...,\}\backslash\{h, k, ...,\}$ is unambiguous.

Given a planar graph G and a particular plane embedding, G^d denotes the *planar dual* of G. $E(G^d) = E(G)$; the labelling and distinguished edge are those of G. $V(G^d)$ is the set of regions into which the embedding of G divides the plane. $e \in E(G^d)$ is incident in G^d with the regions that lie on either side of the path representing e in the embedding of G (these regions may be the same). ε_0 and ε_1 are the regions on either side of the path representing ε.

Series–parallel graphs. Let G, H be graphs with distinguished edges and vertices. To form the *parallel connection* $G \wedge H$, first form a disjoint union $G \cup H$. Then identify ε_{G0} with ε_{H0} to form the distinguished vertex ε_0 in $G \wedge H$. Identify ε_{G1} and ε_{H1} to form ε_1. Finally, remove ε_G and ε_H and replace them by a new distinguished edge ε incident with ε_0 and ε_1.

To form the *series connection* $G \vee H$, again form first a disjoint union $G \cup H$. Identify ε_{G1} with ε_{H0} and let ε_{G0} and ε_{H1} become the distinguished vertices of $G \vee H$. Again remove ε_G and ε_H and replace them by a new distinguished edge ε incident with $\varepsilon_0 = \varepsilon_{G0}$ and $\varepsilon_1 = \varepsilon_{H1}$.

If $b \in \mathscr{A}$, the *atomic series–parallel graph* $\mathbf{b}$ has $V(\mathbf{b}) = \{\varepsilon_0, \varepsilon_1\}$, $E(\mathbf{b}) = \{\varepsilon, e\}$, $l(e) = b$, and $i(b) = E(\mathbf{b}) \times V(\mathbf{b})$ so that both ε and e are incident with ε_0 and ε_1.

A *series–parallel graph* is defined recursively to be either

(i) an atomic series–parallel graph $\mathbf{b}$,

(ii) a parallel connection $G \wedge H$ of series–parallel graphs G, H, or

(iii) a series connection $G \vee H$ of series–parallel graphs G, H.

Let P be a lattice polynomial. Interpreting $\wedge$, $\vee$, a, b, c,..., as $\vee$, $\wedge$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,..., P defines a series–parallel graph G labelled with the variables occurring in P. G is *the series–parallel graph associated to* P.

For an extended discussion of series–parallel graphs and matroids, including classical characterizations, various interesting properties, and (in Brylawski) an elegant formulation in the category of matroids-with-base-point and strong maps, we refer the reader to [8] and [2]. Among the properties we shall need are:

(i) Every series–parallel graph is planar.

(ii) $\mathbf{b}^d \cong \mathbf{b}$, $(G \wedge H)^d \cong G^d \vee H^d$, and $(G \vee H)^d \cong G^d \wedge H^d$ in suitable plane embeddings.

(iii) Series–parallel graps G, H have $G^d \cong H^d$ in some embedding iff G and H are associated to polynomials defining the same element of the free algebra with commutative and associative operations $\wedge$ and $\vee$. G and H with the above properties are said to be *equivalent* series–parallel graphs.

Graphs as relational operators. Let G be a graph with distinguished edge and vertices ε, ε_0, ε_1. Suppose given an interpretation $j: \mathscr{A} \to 2^{S \times S}$ assigning to each label symbol $a \in \mathscr{A}$ a reflexive and symmetric relation $j(a)$ on a set S. Then there is a reflexive relation $j(G)$ on S defined by

$x \sim y[j(G)]$ if there is a function $f: V(G) \to S$ such that $f(\varepsilon_0) = x, f(\varepsilon_1) = y$, and for $e \in E(G) \backslash \varepsilon$ incident with u, $v \in V(G)$,

$$f(u) \sim f(v)[j(l(e))].$$

It follows from this definition that $j(\mathbf{a}) = j(a)$, $j(G \wedge H) = j(G) \cap j(H)$, and $j(G \vee H) = j(G) \circ j(H)$. Consequently, we have the following lemma.

LEMMA 1.1. *Let $j: \mathscr{A} \to L$ be an interpretation of the alphabet into a* *linear lattice L represented on a set S. Let P be a lattice polynomial and G its associated series–parallel graph. Then for x,* $y \in S$,

$$x \sim y[j(P)] \quad \text{if and only if } x \sim y[j(G)].$$

1.0. Proof Theory

The proof theory we present in this section addresses the *generalized word problem* for linear lattices: namely, is a lattice equation $P \leqslant Q$ a consequence for all linear latices of a set of assumptions $\{P_i \leqslant Q_i \mid i \in I\}$? In other words, we are interested in the valid (infinitary) universal Horn sentences [20, 28]

$$\forall a \forall b \cdots (\mathop{\&}_{i \in I} P_i \leqslant Q_i \Rightarrow P \leqslant Q) \tag{1.1}$$

in the theory of linear lattices.

A *proof-theoretic* approach to the word problem for a class of lattices means one modelled on the proof theory for classical propositional logic, viewed as a solution to the word problem for distributive lattices. In this classical theory, we read $\wedge$, $\vee$, and $\leqslant$ as *and, or,* and *implies*, and seek to establish (1.1) as a theorem of logic (equivalently, of distributive lattices) by deducing Q from P according to *rules of inference* which specify the possible deductions, some absolute (from A and B deduce $A \wedge B$) and some relative (from P_i deduce Q_i) [13, 31, 32].

For non-distributive classes of lattices, there may be non-classical rules of inference which are correct and complete for the universal Horn (or sometimes just equational) theories of these classes. Such proof theories can be found for instance in [29] for general lattices and orthocomplemented lattices, and in [23] for n-permutable congruence lattices on universal algebras, including as special cases modular lattices (3-permutable congruences on sets) and linear lattices (2-permutable congruences on sets). Czedli [5] has described an approach to the general lattice word problem that can be construed as a graphical proof theory, a quality it shares with the present work.

Our purpose is to describe a new proof theory for the class of linear lattices. Our theory departs in format from those just mentioned in that instead of anything resembling "well-formed formulas" in the classical sense, the deductions of our theory apply to graphs. This change of viewpoint gains us access to topological graph–theoretic notions which we exploit to get a partial duality result for our proof theory, and, in the second half of the paper, a Normal Form theorem, or "Hauptsatz," for its proofs.

THEOREM 1.1 (MAIN THEOREM ON PROOF THEORY). *The universal Horn sentence* (1.1) *is valid in all linear lattices if and only if the series–parallel graph associated to Q can be obtained from that associated to P via a finite sequence of deductions from the list which follows. Any lattice satisfying all the valid Horn sentences* (1.1) *is linear. The list of deductions* (*which are schematically illustrated in Fig.* 1) *is*:

(A) *Parallel* duplication *of an edge. If $e \neq \varepsilon$ is an edge with label $l(e) = a$, replace it by new edges e', e'' with $l(e) = l(e'') = a$, making e' and e'' incident with the same vertex or vertices as e was.*

(B) Coalescence *of series edges. Let $e \neq \varepsilon$ and $f \neq \varepsilon$ be edges with label $l(e) = l(f) = a$. Suppose that e is incident with vertices u, v; f is incident with v, w; and no other edge is incident with v. Then remove e, f, and v and replace them with a new edge e' with $l(e') = a$, making e' incident with u, w.*

(C) Uncontraction *of an edge. Let G be the graph present before the deduction. Pass to a graph H containing a new edge e with any label $a \in \mathscr{A}$, such that $G = H/e$.*

(D) Deletion *of an edge. Delete any edge $e \neq \varepsilon$.*

(E) Reversal *of a series–parallel subgraph. Let J be a subgraph of the graph G present before the deduction. Suppose J is attached to the rest of G at only two vertices u_0, u_1. That is, u_0 and u_1 are the only vertices incident with edges of both J and $G \backslash J$. Suppose further that $\varepsilon \notin J$ and that upon deleting $G \backslash J$ and replacing it by a new distinguished edge ε with $\varepsilon_0 = u_0$, $\varepsilon_1 = u_1$, a series–parallel graph J' results. Then make each $e \in E(J)$ that is*

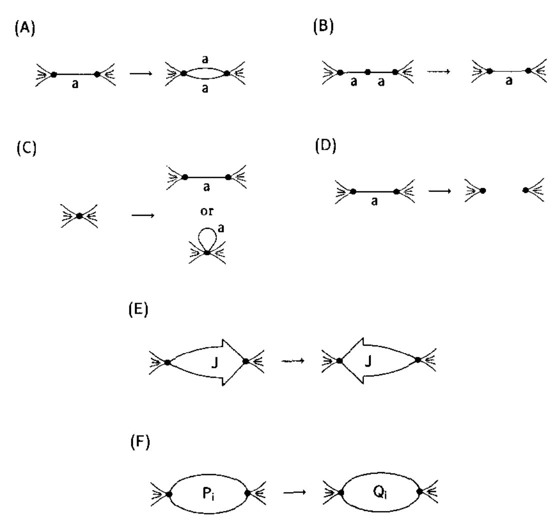

FIG. 1. The deductions.

incident in G with u_i incident instead with u_{1-i} $(i=0, 1)$. The effect of this is to detach J and reattach it with the ends exchanged.

(F) Substitution *of Q_i for P_i. Let J be as in* (E) *and suppose J' is the series–parallel graph associated to P_i. Then remove J and replace it with a new subgraph K attached as J was and for which K' is the series–parallel graph associated to Q_i.*

We also allow passage from a graph to another isomorphic to it as part of a derivation, without explicitly considering this a deduction.

Proof outline of Theorem 1.1. As the proof of Theorem 1.1 is fairly intricate, we outline it here before beginning it. Let us from now on refer to a derivation of Q from P via deductions (A)–(F) as a *proof* of (1.1). Our first step is to verify that (1.1) is indeed valid if it has a proof. That being done, we define *transitive*, *commutative*, and *conditional* extensions of a graph. In Lemmas 1.2 and 1.3 we show how each can be constructed using deductions (A)–(F). From these extensions we build representations of "locally free" linear lattices satisfying $P_i \leqslant Q_i$ for all $i \in I$. In Lemma 1.4, we set forth the properties of these lattices and show that if a certain one satisfies $P \leqslant Q$, then (1.1) has a proof. Finally, given a lattice L satisfying all linearly valid implications (1.1) we combine "locally free" linear lattices so as to construct a linear representation of L from a presentation of L. We remark that the linearity of any lattice satisfying all the valid Horn sentences (1.1) follows from [25], but we include the proof here as it comes almost for free once the rest is done.

Proof of Theorem 1.1. Let L be a linear lattice represented on a set S. Fix an interpretation $j\colon \mathscr{A} \to L$, and suppose that $P_i \leqslant Q_i$ is satisfied under j for each $i \in I$. Let $s_0, s_1 \in S$ have

$$s_0 \sim s_1 [j(P)].$$

Let G be any graph derivable from the series–parallel graph associated to P by deductions (A)–(F). We claim that $s_0 \sim s_1 [j(G)]$. Indeed, when G is the series–parallel graph associated to P, this is true by Lemma 1.1. The condition $s_0 \sim s_1 [j(G)]$ is unaffected by deduction (A) applied to G, and only weakened by deduction (D). It is preserved by (C) because the relations $j(a)$ are reflexive. It is preserved by (B) because the $j(a)$ are transitive. Preservation by (E) follows from Lemma 1.1 applied to a lattice polynomial R with associated series–parallel graph J', where J is the subgraph reversed, and the fact that $j(R)$ is a symmetric relation. Preservation by (F) follows from Lemma 1.1 applied to P_i and Q_i and the fact that L satisfies $P_i \leqslant Q_i$ under the interpretation j.

If (1.1) has a proof, therefore, we have $s_0 \sim s_1[j(H)]$ where H is the series–parallel graph associated to Q, so

$$s_0 \sim s_1[j(Q)]$$

by Lemma 1.1. This shows that L satisfies $P \leqslant Q$ under j and the implication (1.1) is valid.

Let G be a graph. For each $a \in \mathscr{A}$, the edges of G define a reflexive and symmetric relation $k(a)$ on $V(G)$ by

$$s \sim t[k(a)] \qquad \text{if } s = t \text{ or } \exists e \in E(G) \text{ incident with } s \text{ and } t, \text{ and } l(e) = a.$$

If J is another graph, with distinguished edge and vertices, the interpretation $k: \mathscr{A} \to 2^{V(G) \times V(G)}$ induces a relation $k(J)$ on $V(G)$ as in the remarks before Lemma 1.1 in the preliminaries.

We now define transitive, commutative, and conditional *extensions* of a graph. These form the building blocks for the coming construction of "locally free" linear lattices.

For each extension, we begin with a graph G and vertices $u_0, u_1 \in V(G)$ with $u_0 \sim u_1[k(J)]$, where J has a specified form. To make the extension, we take a graph N (with distinguished edge and vertices), depending on J, and attach it to G at u_0 and u_1. More precisely, we form a disjoint union $G \cup (N \backslash \varepsilon)$, then identify u_0 with ε_{N0} and u_1 with ε_{N1}. Note that if $u_0 = \varepsilon_{G0}$ and $u_1 = \varepsilon_{G1}$, then the extension is the same as $G \wedge N$. Note also that necessarily $u_0 \sim u_1[k(N)]$ in the extended graph.

When J is of the form $K \vee K$ for a series–parallel graph K, take $N = K$. Then the graph formed as above is a *transitive extension* of G.

When J is of the form $K \vee H$ for series–parallel graphs K, H, take $N = H \vee K$. Note that N is attached so that H and K are oriented oppositely to their "images" in G. In this case, the graph formed as above is a *commutative extension* of G.

Finally, when J is the series–parallel graph associated to P_i, take N to be the series–parallel graph associated to Q_i. In this last case, the graph formed as above is a *conditional extension* of G *relative to* $P_i \leqslant Q_i$.

The three types of extensions are illustrated schematically in Fig. 2.

An extension is determined up to isomorphism by its type (transitive, commutative, or conditional, and if conditional, relative to which hypothesis $P_i \leqslant Q_i$) and by u_0, u_1, J, and N. Thus we may speak of "all" extensions of G, by which we actually mean a set consisting of one representative of each isomorphism class. Let us take the new subgraphs N added to G by different extensions to be disjoint, so that we can refer unambiguously to the union of extensions.

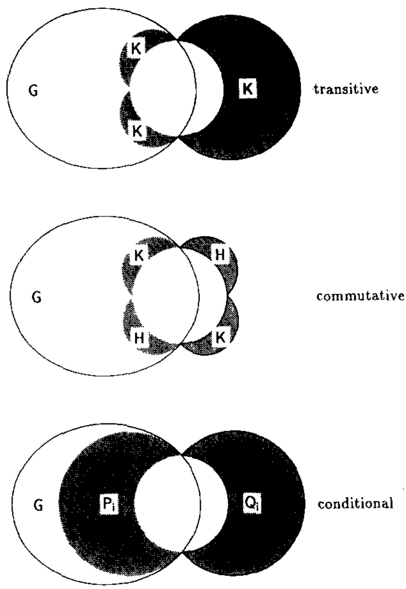

FIG. 2. The three extensions of a graph

It proves convenient to define one more extension. When J is any finite not necessarily series–parallel graph with distinguished edge and vertices, take $N = J$ and call the graph formed by attaching N a *plain extension* of G.

LEMMA 1.2. *Any plain extension of G can be derived from G using deductions* (A), (C), *and* (D).

Proof. Let u_0, u_1, J, $N = J$ define the extension and let H be the resulting graph. Recall that $u_0 \sim u_1[k(J)]$ means there is a function f: $V(J) \to V(G)$ such that $f(\varepsilon_0) = u_0$, $f(\varepsilon_1) = u_1$, and whenever $e \in E(J) \backslash \varepsilon$ is incident with v, $w \in V(J)$, either $f(v) = f(w)$ or $\exists g_e \in E(G)$ incident with $f(v)$ and $f(w)$, such that $l(g_e) = l(e)$.

We now define some graphs. For every $v \in V(J) \backslash \{\varepsilon_0, \varepsilon_1\}$, add an edge e_v to H incident with v and $f(v)$. Call the resulting graph I. Contract all the edges e_v to get $K = I/\{e_v\}$. Some edges $e \in E(J) \backslash \varepsilon \subseteq E(H) = E(K)$ may be loops in K. Contract them to get $L = K/\{\text{loops of } J\}$. Now $V(L) = V(G)$ as each vertex $v \in V(J) \backslash \{\varepsilon_0, \varepsilon_1\} = V(H) \backslash V(G)$ has been identified in L with $f(v) \in V(G)$. Also $E(G) \subseteq E(L)$ and each $e \in E(L) \backslash E(G) \subseteq E(J)$ is parallel to (i.e., incident with the same vertices as) g_e. Since $l(e) = l(g_e)$, L can be obtained from G by applying deduction (A) once for each $e \in E(L) \backslash E(G)$. L is a contraction of I by a finite set of edges, so I can be obtained from L

by repeated application of (C). Finally, H can be obtained from I by using deduction (D) to delete each edge e_v. ∎

LEMMA 1.3. *Any transitive, commutative, or conditional extension of G can be derived from G using deductions* (A)–(F).

Proof. For a commutative extension, use Lemma 1.2. to make a plain extension by $K \vee H$. Three applications of (E)—one to turn the whole $K \vee H$ around and two to point the subgraphs K and H back in the original direction—convert the attached $K \vee H$ to an attached $H \vee K$.

Similarly, a plain extension followed by an application of (F) realizes conditional extensions.

We now show how to convert an attached $K \vee K$ into an attached K by applications of (B), (C), (D), and (E), realizing transitive extensions and completing the proof of the Lemma. If K is atomic, one application of (B) works. Otherwise, suppose by induction that $H \vee H$ can be converted to H for each proper series–parallel subgraph H of K. If $K = A \vee B$ is a series connection, then (E) converts $K \vee K = A \vee B \vee A \vee B$ into $A \vee A \vee B \vee B$. Separate conversions of $A \vee A$ to A and $B \vee B$ to B convert this to $A \vee B = K$. If $A \wedge B$ is a parallel connection, let u be the common vertex of the two subgraphs $A \wedge B$ of $K \vee K = (A \vee B) \vee (A \wedge B)$. Using (C), uncontract an edge z at u, partitioning u into vertices u' and u'', both incident with z, so that edges from subgraphs A are incident with u' and edges from subgraphs B are incident with u''. Then delete z, using (D). This leaves $(A \vee A) \wedge (B \vee B)$, from which $A \vee A$ and $B \vee B$ can be separately converted to A and B to get $A \wedge B = K$. ∎

Using one last Lemma, we can now complete the proof of Theorem 1.1. Consider again the implication (1.1). We define an infinite sequence of graphs

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots.$$

G_0 is the series–parallel graph associated to P. For $m > 0$, G_m is inductively defined to be the union of all extensions (transitive, commutative, or conditional relative to $P_i \leqslant Q_i$ in (1.1)) of G_{m-1}. We set also

$$G_\infty = \bigcup_{m \geqslant 0} G_m.$$

Recall from the remarks preceding the definition of the extensions the interpretation k of the alphabet $\mathscr{A}$ by relations on $V(G_\infty)$:

$$s \sim t[k(b)] \qquad (s, t \in V(G_\infty))$$

if either $s = t$, or there is an edge $g \in E(G_\infty)$ with $l(g) = b$ which is incident with s and t.

LEMMA 1.4. *The relations $k(b)$, $b \in \mathscr{A}$, generate a linear lattice L_1 represented on the set $V(G_\infty)$. L_1 satisfies $P_i \leqslant Q_i$ for all $i \in I$. If L_1 satisfies $P \leqslant Q$ under k then the series–parallel graph associated to Q is derivable from G_0 by deductions* (A)–(F).

Proof. Recall that for each series–parallel graph T the interpretation k induces a reflexive relation $k(T)$ on $V(G_\infty)$ such that

$$\begin{aligned} k(\mathbf{b}) &= k(b) \qquad \text{for each variable } b,\\ k(U \wedge V) &= k(U) \cap k(V),\\ k(U \vee V) &= k(U) \circ k(V), \end{aligned} \tag{1.2}$$

and for $s, t \in V(G_\infty)$,

$$s \sim t[k(T)]$$

iff there is a function f: $V(T) \to V(G_\infty)$ such that $f(\varepsilon_0) = s$, $f(\varepsilon_1) = t$, and for $e \in E(T) \backslash \varepsilon$ incident with v, $w \in V(T)$, either $f(v) = f(w)$ or $f(v) \sim f(w)$ $[k(l(e))]$.

Suppose for some s, $t \in V(G_\infty)$ that

$$s \sim t[k(U) \circ k(V) = k(U \vee V)].$$

This says there is a function f: $V(U \vee V) \to V(G_\infty)$ with the above properties. Since $U \vee V$ is finite, the conditions on f are already satisfied in some G_m and $s \sim t[k(U \vee V)]$ in G_m. Since G_{m+1} contains every commutative extension of G_m, it follows that in G_{m+1}, hence in G_∞,

$$s \sim t[k(V \vee U) = k(V) \circ k(U)].$$

Therefore the relations $k(T)$ commute. Using this fact, an induction on (1.2) shows the $k(T)$ are symmetric.

Suppose

$$s \sim t[k(U) \circ k(U) = k(U \vee U)].$$

Then there is a function f: $V(U \vee U) \to V(G_\infty)$ with the properties specified above. Since $U \vee U$ is finite, the conditions on f are already satisfied in some G_m. Since G_{m+1} contains every transitive extension of G, it follows that in G_{m+1}, hence in G_∞,

$$s \sim t[k(U)]$$

and the relations $k(T)$ are transitive. Hence the $k(T)$ are commuting equivalence relations, so

$$k(U) \circ k(V) = k(U) \vee k(V).$$

From this it follows by induction on (1.2) that $k(R)=k(\mathbf{R})$, where R is a lattice polynomial and $\mathbf{R}$ its associated series–parallel graph. This proves the first assertion of the Lemma.

Applying once again the argument used to show that the $k(T)$ commute and are transitive, but working now with conditional extensions, we obtain the second assertion, that L_1 satisfies $P_i \leqslant Q_i$ for each $i \in I$.

For the last assertion, note that in the representation of L_1 on $V(G_\infty)$, $\varepsilon_0 \sim \varepsilon_1\ [k(P)]$ by Lemma 1.1, the definition of k, and the fact that $G_0 \subseteq G_\infty$ is the series–parallel graph associated to P. If L_1 satisfies $P \leqslant Q$ under the interpretation k then $\varepsilon_0 \sim \varepsilon_1 [k(Q)]$. Let H be the series–parallel graph associated to Q. By Lemma 1.1 there is a function $f: V(H) \to V(G_\infty)$ such that $f(\varepsilon_{H0}) = \varepsilon_0$, $f(\varepsilon_{H1}) = \varepsilon_1$, and for $e \in E(H)\backslash \varepsilon$ incident with v, $w \in V(H)$, either $f(v) = f(w)$ or $f(v)$ and $f(w)$ are incident with an edge $g_e \in E(G_\infty)$ having $l(g_e) = l(e)$. These conditions on f depend only on finitely many edges of $E(G_\infty)$, so are already satisfied in a finite subgraph $N \subseteq G_\infty$. We show that N can be derived from G_0 by a finite sequence of deductions (A)–(F). By Lemma 1.2, another finite sequence of deductions then realizes the plain extension of N by H attached at ε_0, ε_1. Finally, applications of (D) remove the original edges of N, leaving behind H, the series–parallel graph associated to Q, and proving the last assertion of the Lemma.

That N is derivable from G_0 by a finite sequence of deductions follows from Lemma 1.3 and the observation that any finite subgraph $N \subseteq G_\infty$ is contained in a subgraph of G_∞ obtainable from G_0 by finitely many transitive, commutative, and conditional extensions.

To verify this observation, note that $N \subseteq G_m$ for some m because N is finite. If $m=0$, the observation is trivial. By induction, assume the observation is true for all finite subgraphs of G_{m-1}, when $m>0$. G_m is the union of all extensions of G_{m-1}, so N, being finite, is contained in the union of finitely many of them, say

$$N \subseteq \bigcup_{j=1}^{n} M_j,$$

where each M_j is an extension (determined, say, by u_{j0}, u_{j1}, J_j, and N_j) of G_{m-1}. The conditions for the extension depend only on finitely many edges of G_{m-1}, so are already satisfied in some finite subgraph $T_j \subseteq G_{m-1}$. Define

$$N' = (N \cap G_{m-1}) \cup \bigcup_{j=1}^{n} T_j.$$

N' is a finite subgaph of G_{m-1}. By induction we can find $N^+ \subseteq G_{m-1}$ containing N' such that N^+ is obtainable from G_0 by a finite sequence of extensions. Let M_j^+ consist of N^+ together with the new part of the extension M_j of G_{m-1}. Since $T_j \subseteq N^+$, M_j^+ is an extension of N^+. Furthermore,

$$N \subseteq \bigcup_{j=1}^{n} M_j^+$$

so the union on the right-hand side is the required subgraph.

Suppose now that (1.1) holds in every linear lattice. Then in particular, it holds in L_1 and by Lemma 1.4, (1.1) has a proof.

Since we have already shown that if (1.1) has a proof then it holds in every linear lattice, Lemma 1.4 tells us that as Q varies, L_1 satisfies exactly those inequalities $P \leqslant Q$ which are consequences for all linear lattices of the inequalities $P_i \leqslant Q_i$. Were P and Q both allowed to vary, this property would make L_1 a free linear lattice relative to the conditions $P_i \leqslant Q_i$. Since P is fixed for the lattice L_1, we prefer to say that it is "locally free" at P.

Finally, let L be an arbitrary lattice satisfying every implication (1.1) that has a proof. Select a set of elements generating L and for the moment take the alphabet $\mathscr{A}$ to be this set. Define $\{P_i \leqslant Q_i \mid i \in I\}$ to be the set of all polynomial inequalities in the generators that hold in L. For every lattice polynomial P in the generators, construct $G_\infty(P)$ as above. Lemma 1.4 gives a lattice $L_1(P)$ represented on $V(G_\infty(P))$ and since $L_1(P)$ satisfies $P_i \leqslant Q_i$ for all $i \in I$, there is a canonical homomorphism

$$\rho_P : L \rightarrow L_1(P)$$

which provides a not necessarily faithful linear representation of L on $V(G_\infty(P))$. Form the sum of these representations for all P to get a representation of L on the disjoint union

$$S = \bigcup_P V(G_\infty(P))$$

This representation is faithful: for suppose $p \nleqslant q$ for $p, q \in L$. Let $p = P$, $q = Q$ be expressions of p and q as lattice polynomials in the generators. By assumption,

$$P_i \leqslant Q_i \qquad \text{for all } i \in I \tag{1.3}$$

implies

$$P \leqslant Q \tag{1.4}$$

cannot have a proof since L satisfies (1.3) but not (1.4) under the canonical interpretation of the elements of $\mathscr{A}$ (which are generators of L) as themselves. By Lemma 1.4, then, $P \leqslant Q$ is not satisfied in $L_1(P)$ and so there are vertices s, t (in fact, ε_0, ε_1) in $V(G_\infty(P))$, hence in S, such that

$$s \sim t[p]; \qquad s \not\sim t[q].$$

This completes the proof of Theorem 1.1. ∎

Remarks on duality. Recall that the *dual* of a lattice L is the lattice L^d on the same underlying set as L, but in which meet and join have been interchanged, or what is the same, the partial order reversed. As is well known, the varieties of distributive lattices, modular lattices, and all lattices are closed under dualization. Jónsson [26] showed that the variety of Arguesian lattices is also self-dual in this sense (see (2.2))

While it remains an open question whether the class of all linear lattices is self-dual, there is an intrinsic duality to the list of deductions (A)–(F). Under certain circumstances, we can exploit this duality to obtain from a proof of (1.1) a proof of the dual implication. We now explain how this can be done.

The dual implication to (1.1) is the implication

$$\forall a \forall b \cdots (\mathop{\&}_{i \in I} Q_i^d \leqslant P_i^d \Rightarrow Q^d \leqslant P^d). \tag{1.5}$$

It follows from the planar duality of $\wedge$ and $\vee$ mentioned in the preliminaries that the series–parallel graph associated to P^d is planar dual to that associated to P.

Suppose (1.1) has a proof in which each intermediate graph appearing in the derivation of Q from P is planar. Suppose further that we can select plane embeddings of these graphs in such a way that each deduction "takes place in the plane." This requirement has a precise meaning. For all deductions but (C), it is that (i) only the plane embedding of the part of the graph replaced or removed by the deduction is changed, and (ii) the edges removed from the old graph and those introduced in the new graph are embedded within a simply connected region containing no other part of either graph. For deduction (C), the meaning is that the contraction from the new graph back to the old can be realized by a continuous deformation of the plane. Under these hypotheses, we say that (1.1) has a *planar* proof.

COROLLARY 1.1 (PROOF DUALITY). *If* (1.1) *has a planar proof, the dual implication* (1.5) *has a proof.*

Proof. Let $G_0, G_1, \ldots, G_n$ be the sequence of intermediate plane graphs occurring in the planar proof of (1.1). G_0 and G_n are the series–parallel networks associated to P and Q, respectively, so G_n^d and G_0^d are associated to Q^d and P^d or to polynomials equivalent to them up to associativity and commutativity of $\wedge$ and $\vee$.

We claim that $G_n^d, G_{n-1}^d, \ldots, G_0^d$ is the sequence of intermediate graphs in a proof of (1.5). In fact, it is easy to check that if G_{j+1} is derived from G_j by deduction (A), (B), (C), or (D), taking place in the plane, then G_j^d can be derived from G_{j+1}^d by deduction (B), (A), (D), or (C), respectively. If G_{j+1} is derived from G_j by deduction (E), then G_{j+1}^d can be derived from G_j^d by zero or more applications of (E), depending upon how the reversed subgraph is embedded. Finally, if G_{j+1} is derived from G_j by deduction (F) using $P_i \leqslant Q_i$, then G_j^d can be derived from G_{j+1}^d by (F) using $Q_i^d \leqslant P_i^d$, together with some applications of (E) if necessary to adjust the embedding. ∎

EXAMPLE 1.1. The *Arguesian implication* [24, 36] is

$$(a \vee a') \wedge (b \vee b') \leqslant c \vee c' \tag{1.6a}$$

implies

$$(a \vee b) \wedge (a' \vee b') \leqslant [(a \vee c) \wedge (a' \vee c')] \vee [(c \vee b) \wedge (c' \vee b')]. \tag{1.6b}$$

If a, b, c, a', b', c' are points in a projective plane (i.e., atoms in its lattice of flats), then (1.6a) expresses that lines aa', bb', and cc' are concurrent, or that triangles abc and $a'b'c'$ are "centrally perspective." (1.6b) expresses that points $ab \cap a'b'$, $bc \cap b'c'$, and $ac \cap a'c'$ are collinear, or that abc and $a'b'c'$ are "axially perspective." That the latter follows from the former in a coordinatizable projective plane is Desargues' theorem of projective geometry; hence, "Arguesian implication." Figure 3 shows the plane configuration just described.

A proof of (1.6) is shown in Fig. 4. Series–parallel graphs are shown with their associated lattice polynomials. The sequence of deductions is detailed at the bottom. The proof shown is planar. The planar dual to each graph is

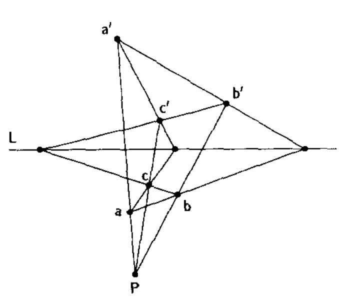

FIG. 3. The Desargues configuration, P and L are the center and axis of perspectivity.

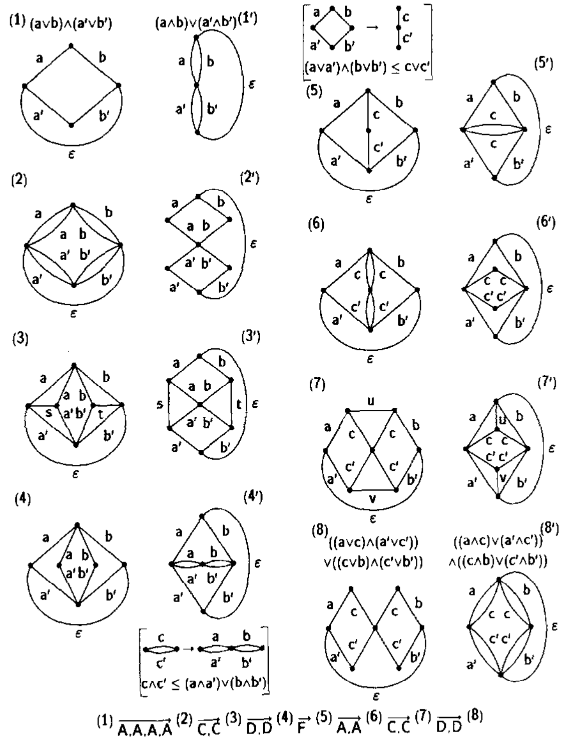

FIG. 4. Proof of Desargues' implication and its dual.

shown beside it with a primed number. Reading the primed graphs backward gives a proof of the dual implication

$$c \wedge c' \leqslant (a \wedge a') \vee (b \wedge b') \tag{1.7a}$$

implies

$$[(a \wedge c) \vee (a' \wedge c')] \wedge [(c \wedge b) \vee (c' \wedge b')] \leqslant (a \wedge b) \vee (a' \wedge b'). \tag{1.7b}$$

In fact, (1.6a, b) and (1.7a, b) are equivalent in any lattice [26], as an easy substitution shows.

Closure properties. We end this section with some remarks on closure properties of the class of linear lattices under various general algebraic constructions. By general theorems of universal algebra ([15], Appendix 4, Theorems 3, 4, for instance), the axiomatizability of the class of linear lattices by universal equational Horn sentences, such as (1.1), is equivalent to the closure of the class of linear lattices under isomorphism, sublattices, and products. These closure properties can also be seen directly. Namely, closure under isomorphism and sublattices are built into the definition, and closure under products is given by the sum of representations construction.

From Theorem 1.1 it is clear that every implication (1.1) depends on only finitely many of its hypotheses $P_i \leqslant Q_i$. This fact implies the further closure of the class of linear lattices under direct limits. It does not seem possible, however, straightforwardly to construct a representation of a direct limit of linear lattices from representations of the individual lattices.

It also does not seem possible, straightforwardly or otherwise, to construct a linear representation of an arbitrary homomorphic image of a linear lattice. To do so, of course, would show that the class of linear lattices is a variety, answering the most important open question concerning linear lattices. Whether linear lattices form a variety or not, one can in any case study the variety they generate, i.e., study the valid linear lattice *identities* as opposed to *implications*. We begin this study in the next section.

2.0. Identities and Normal Form

We turn now from the universal Horn theory of linear lattices to their *equational* theory, i.e., to the *identities*

$$P \leqslant Q \tag{2.1}$$

satisfied in all linear lattices. Since a lattice identity is the same thing as an implication with no hypotheses, the Main Theorem on Proof Theory tells us that (2.1) holds if and only if it has a proof using deductions (A)–(E). Our program is to convert any such proof into one having a canonical form. This form is stated in Theorem 2.2, which is thus a Normal Form Theorem for linear lattice proof theory.

The Normal Form Theorem comes very close to giving a solution of the word problem for free linear lattices. We advance two conjectures, one much stronger than the other, either of which would if true make the Normal Form Theorem a solution of the free linear lattice word problem. This would of course stand in contrast to the situation for free modular lattices, proved in [9] to have unsolvable word problems.

We prove the stronger conjecture for the special case of proofs not using deduction (E), but remark that there are proofs in which use of (E) is essential. We also give an example of a linear lattice identity whose dual fails if the stronger conjecture holds. Thus a proof of the stronger conjecture would have quite far-reaching consequences, as it would show that linear lattices generate a non-self-dual variety, which then could not be equal to the variety of Arguesian lattices.

Our Normal Form Theorem rests upon three main ideas, which we develop separately before stating the theorem. The first idea is to eliminate repeated and unbalanced variables from the identity $P \leqslant Q$. The second is to reduce $P \leqslant Q$ to what we call a "short" identity. The third is to apply certain commutativity relationships which hold among the deductions.

We begin with a result due to G.-C. Rota (unpublished communication).

PROPOSITION 2.1. *Any lattice identity $P \leqslant Q$ is equivalent to one in which every variable appears exactly once on each side.*

Proof. Let x be a variable appearing in $P \leqslant Q$. If x appears only in P, substitute 1 for each occurrence of x. If x appears only in Q, substitute 0 for each occurrence of x. Otherwise, x appears m times in P and n times in Q for some $m, n \geqslant 1$. Introduce new variables x_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$. Replace the ith occurrence of x in P by $x_{i1} \wedge \cdots \wedge x_{in}$ for each $1 \leqslant i \leqslant m$. Dually, replace the jth occurrence of x in Q by $x_{ij} \vee \cdots \vee x_{mj}$.

Let the resulting identity be $P' \leqslant Q'$, and suppose $P' \leqslant Q'$ is valid in a lattice L. If x appeared only in P, then $Q = Q'$ and $P \leqslant P' = P|_{x \leftarrow 1}$ because lattice operations are monotonic. Hence $P \leqslant Q$ is valid in L. If x appeared only in Q, then $P = P'$ and $Q' \leqslant Q$ by monotonicity. Hence again $P \leqslant Q$ is valid in L. If x appeared in both P and Q, substitute x for each x_{ij} in $P' \leqslant Q'$. We then recover $P \leqslant Q$, so again $P \leqslant Q$ is valid in L.

Conversely, suppose $P \leqslant Q$ is valid in L. If x appeared only in P, substitute 1 for x in $P \leqslant Q$. This gives $P' \leqslant Q'$, which is therefore valid in L. If x appeared only in Q, substitute 0 for x in $P \leqslant Q$. This again gives $P' \leqslant Q'$ which is therefore valid in L. If x appeared in both P and Q, substitute

$$\bar{x} = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} x_{ij}$$

for x in $P \leqslant Q$. Now for any particular i_0, j_0 we have

$$x_{i_0 1} \wedge \cdots \wedge x_{i_0 n} \leqslant \bar{x};$$

$$\bar{x} \leqslant \bigwedge_{j=1}^{n} \bigvee_{i=1}^{m} x_{ij} \leqslant x_{1j_0} \vee \cdots \vee x_{mj_0}.$$

$P' \leqslant Q'$ is then valid in L by monotonicity. This shows that $P' \leqslant Q'$ is an equivalent identity to $P \leqslant Q$.

Carrying out the above procedure for each variable appearing in $P \leqslant Q$ and simplifying to eliminate 0's and 1's we obtain the required equivalent identity. ▮

EXAMPLE 2.2. The *Arguesian identity* ([36, 7]—our version is a simplified equivalent form [17]) is

$$c \wedge ([(a \vee a') \wedge (b \vee b')] \vee c')$$
$$\leqslant a \vee ([((a \vee b) \wedge (a' \vee b')) \vee ((b \vee c) \wedge (b' \vee c'))] \wedge (a' \vee c')).$$

We now apply Proposition 2.1. Since every variable already occurs exactly once on the left-hand side, the second subscript on the replacement variables does not vary, and we suppress it. The equivalent form then reads

$$\begin{aligned} & c \wedge ([((a_1 \wedge a_2) \vee (a'_1 \wedge a'_2)) \\ & \vee ((b_1 \wedge b_2) \vee (b'_1 \wedge b'_2))] \vee (c'_1 \wedge c'_2)) \\ & \leqslant a_1 \vee ([((a_2 \vee b_1) \wedge (a'_2 \vee b'_1)) \\ & \vee ((b_2 \vee c) \wedge (b'_2 \vee c'_1))] \wedge (a'_1 \vee c'_2)). \end{aligned} \tag{2.2}$$

It is a remarkable fact that (2.2) becomes its own dual after exchanging the symbols $a_1 \leftrightarrow c$, $a'_1 \leftrightarrow c'_1$, $a_2 \leftrightarrow b_2$, and $a'_2 \leftrightarrow b'_2$. (2.2) is the first explicitly self-dual form of the Arguesian identity to have been found. Jónsson's [26] proof that the variety of Arguesian lattices is self-dual made use of the equivalence of the Arguesian identity to the Arguesian implication (Example 1.1) and the equivalence of that in turn to its dual.

Short identities. Let $P \leqslant Q$ be a lattice identity, valid or not. Let x be a variable occurring in $P \leqslant Q$. Substituting 1 for x in P and 0 for x in Q yields an identity $P' \leqslant Q'$ which is stronger than $P \leqslant Q$ by monotonicity. That is, if $P' \leqslant Q'$ is valid in L, then so is $P \leqslant P' \leqslant Q' \leqslant Q$. If $P' \leqslant Q'$, hence also $P \leqslant Q$, is valid in all linear lattices, we say that $P' \leqslant Q'$ is a *shorter* linear lattice identity than $P \leqslant Q$. A linear lattice identity $P \leqslant Q$ is *short* if there is no shorter valid identity.

Evidently, we lose no generality by confining our attention to short identities. For every linear lattice identity is a consequence of a short identity obtained by shortening it with respect to zero or more of its variables. Therefore, if we have a way to recognize the members of a set of valid identities that includes all the short ones, we can recognize any valid identity by examining all shortenings of it.

Commutativity relationships. Under appropriate conditions, certain sequences of deductions may be replaced by others, without altering the net effect on the graphs on which they act. We call these relationships *commutativity* relationships, and our next step is to exhibit a list of those we need. For this purpose, it is helpful to use a shorthand notation. We give the original sequence of deductions by their letter names, A through E, then an arrow, then the replacement sequence. Conditions under which the replacement is allowed we indicate in parentheses. When we must indicate on what edges a deduction acts, we signify them by symbols in brackets after the deduction name, as follows:

A[e; e', e'']	e is the duplicated edge; e', e'' the duplicates.
B[e, f; e']	e, f are the two series edges; e' is the coalesced edge.
C[e]	e is the uncontracted edge.
D[e]	e is the deleted edge.

LEMMA 2.5.

$$\begin{aligned}
&\mathrm{B}[e,f;e'],\ \mathrm{A}[g;g',g''] \to \mathrm{A, B} && (g \neq e')\\
&\qquad\qquad \to \mathrm{A, A, C, D, B, B} && (g = e')\\
&\mathrm{C}[e],\ \mathrm{A}[f;f',f''] \to \mathrm{A, C} && (e \neq f)\\
&\qquad\qquad \to \mathrm{C, C} && (e = f)\\
&\mathrm{A}[e;e',e''],\ \mathrm{D}[f] \to \mathrm{D, A} && (f \neq e', f \neq e'')\\
&\qquad\qquad \to \varnothing && (f = e' \text{ or } f = e'')\\
&\mathrm{D, A} \to \mathrm{A, D}\\
&\mathrm{E, A} \to \mathrm{A, E}\\
&\mathrm{B, C} \to \mathrm{C, B}\\
&\mathrm{C}[e],\ \mathrm{B}[f,g;f'] \to \mathrm{B, C} && (e \neq f, e \neq g)\\
&\qquad\qquad \to \varnothing && (e = f \text{ or } e = g)\\
&\mathrm{B}[e,f;e'],\ \mathrm{D}[g] \to \mathrm{D, B} && (g \neq e')\\
&\qquad\qquad \to \mathrm{D, D} && (g = e')\\
&\mathrm{B, E} \to \mathrm{E, B}\\
&\mathrm{D, C} \to \mathrm{C, D}\\
&\mathrm{C}[e],\ \mathrm{D}[f] \to \mathrm{D, C} && (e \neq f)\\
&\mathrm{C, E} \to \mathrm{E, C}\\
&\mathrm{E, D} \to \mathrm{D, E}\\
&\mathrm{C}[e],\ \mathrm{C}[f] \to \mathrm{C}[f],\ \mathrm{C}[e]\\
&\mathrm{D}[e],\ \mathrm{D}[f] \to \mathrm{D}[f],\ \mathrm{D}[e].
\end{aligned}$$

Proof. None of the verifications is difficult. We write out only some of the less obvious ones.

$$\mathrm{B}[e,f;e'],\ \mathrm{A}[g;g',g''] \to \mathrm{A, A, C, D, B, B} \qquad (g = e'):$$

The original deductions replace the series pair e, f by the parallel pair g, g', all four edges having a common label $l(e)=l(f)=l(g)=l(g')=a$. Let u be the vertex incident with both edges of the series pair e, f. Apply (A) twice to duplicate e and f, getting e', e'' and f', f''. u is now incident with the edges e', e'', f', f'' and no others. Using (C), uncontract an edge h at u, partitioning u into u' incident with e', f', h, and u'' incident with e'', f'', h. Using (D), delete h. e', f' and e'', f'' are now series pairs. Apply (B) to coalesce each, constructing the required graph.

E, A $\rightarrow$ A, E:

Let J be the subgraph reversed by (E) and e the edge duplicated by (A). The order of the duplication and reversal obviously has no effect on the final graph. The only problem is that applying (A) first might interfere with the preconditions for applying (E). But these preconditions are only that J is series–parallel and attached at two vertices. Duplicating an edge, whether in J or not, affects neither condition.

E, D $\rightarrow$ D, E:

Again, the only problem is that applying (D) first might interfere with the preconditions for applying (E). To prevent this, we must allow a possibly non-2-connected subgraph of a series–parallel graph to be considered series–parallel. This is a harmless technical modification which does not change what can be proved according to the Main Theorem on Proof Theory. Therefore we accept this modification from now on.

C[e], C[f] $\rightarrow$ C[f], C[e]:

Let the left-hand side take us from a graph G to a graph H. Then $G \cong H/\{e,f\} \cong (H/e)/f$. Using the deductions on the right-hand side, we can go from G to H/e to H.

The rest of the verifications are straightforward and left to the reader. ∎

THEOREM 2.2 (NORMAL FORM THEOREM). *Let $P \leqslant Q$ be a short identity valid in linear lattices, in which every variable appears exactly once on each side. Then $P \leqslant Q$ has a proof in which*:

(I) *All deductions* (A) *occur at the beginning.*

(II) *All deductions* (B) *occur at the end.*

(III) *All deductions* (C) *and* (D) *occur in consecutive pairs* (C), (D) *in which the edge uncontracted is immediately deleted. Thus* (C) *and* (D) *can be replaced by a single deduction* (CD) *which consists of partitioning a vertex.*

Proof. Since $P \leqslant Q$ is valid, it has a proof $\mathscr{D}$. We first show how to convert $\mathscr{D}$ into a proof satisfying (I) and (II). As long as $\mathscr{D}$ does not satisfy (I), at least one of the rules

$$\begin{aligned} &\mathrm{B}[e, f; e'], \mathrm{A}[g; g', g''] \to \mathrm{A}, \mathrm{B} && (g \neq e') \\ &\qquad\qquad\qquad\qquad \to \mathrm{A}, \mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{B}, \mathrm{B} && (g = e') \\ &\mathrm{C}[e], \mathrm{A}[f; f', f''] \to \mathrm{A}, \mathrm{C} \quad (e \neq f) \\ &\qquad\qquad\qquad \to \mathrm{C}, \mathrm{C} \quad (e = f) \\ &\mathrm{D}, \mathrm{A} \to \mathrm{A}, \mathrm{D} \\ &\mathrm{E}, \mathrm{A} \to \mathrm{A}, \mathrm{E} \end{aligned} \tag{2.3}$$

may be applied to it. Any proof $\mathscr{D}$ consists of a finite sequence of deductions. Let $S_1, ..., S_k$ be the maximal subsequences consisting of consecutive deductions other than (A), and preceding some instance of (A). Thus any deductions after the last occurrence of (A) are not included in the S_i. Let $|S_i|$ be the length of S_i, and set

$$p(\mathscr{D}) = \sum_{i=1}^{k} 5^{k-i} |S_i|.$$

Evidently, $p(\mathscr{D}) \geqslant 0$ with equality if and only if $\mathscr{D}$ satisfies (I). Furthermore, applying any of the rules (2.3) decreases $p(\mathscr{D})$ by at least 1. Note that the factor 5^{k-i} assures this even for the rule

$$\mathrm{B}, \mathrm{A} \to \mathrm{A}, \mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{B}, \mathrm{B}.$$

Hence $\mathscr{D}$ can be converted into a proof satisfying (I) by at most $p(\mathscr{D})$ applications of the rules (2.3).

As long as $\mathscr{D}$ satisfies (I) but not (II), at least one of the rules

$$\begin{aligned} &\mathrm{B}, \mathrm{C} \to \mathrm{C}, \mathrm{B} \\ &\mathrm{B}[e, f; e'], \mathrm{D}[g] \to \mathrm{D}, \mathrm{B} \quad (g \neq e') \\ &\qquad\qquad\qquad\quad \to \mathrm{D}, \mathrm{D} \quad (g = e') \\ &\mathrm{B}, \mathrm{E} \to \mathrm{E}, \mathrm{B} \end{aligned} \tag{2.4}$$

may be applied to it, without affecting (I). Let $T_1, ..., T_k$ be the maximal subsequences consisting of consecutive deductions in $\mathscr{D}$ other than (B), and following some instance of (B). Thus any deductions before the first occurrence of (B) are not included in the T_i. Set

$$r(\mathscr{D}) = \sum_{i=1}^{k} i\, |T_i|.$$

Evidently, $r(\mathscr{D})\geqslant 0$ with equality if and only if $\mathscr{D}$ satisfies (II). Since each of the rules (2.4) decreases $r(\mathscr{D})$ by at least 1, $\mathscr{D}$ can be converted into a proof satisfying (I) and (II) by at most $r(\mathscr{D})$ applications of these rules.

Therefore let $\mathscr{D}$ be a proof of $P\leqslant Q$ satisfying (I) and (II). Suppose e is an edge introduced by a deduction (C) and later coalesced with another edge f by a deduction (B). Applying the rules

$$\begin{array}{ll} \mathrm{C}[e], \mathrm{B}[g, h; g'] \to \mathrm{B}, \mathrm{C} & (e\neq g, e\neq h) \\ \mathrm{C}[e], \mathrm{C}[g] \to \mathrm{C}[g], \mathrm{C}[e] & \\ \mathrm{C}[e], \mathrm{D}[g] \to \mathrm{D}, \mathrm{C} & (e\neq g) \\ \mathrm{C}, \mathrm{E} \to \mathrm{E}, \mathrm{C} & \end{array}$$

we can assume that e is uncontracted and then immediately coalesced. By the rule

$$\mathrm{C}[e], \mathrm{B}[f, g; f'] \to \varnothing \qquad (e=f \text{ or } e=g)$$

we get a proof in which e is never introduced at all, and the total number of instances of (C) has been decreased. Therefore we may assume no edge is uncontracted using (C) and later coalesced using (B).

By a completely symmetric argument, using the rules

$$\begin{array}{ll} \mathrm{A}[e; e', e''], \mathrm{D}[f] \to \mathrm{D}, \mathrm{A} & (f\neq e', f\neq e'') \\ \mathrm{D}[e], \mathrm{D}[f] \to \mathrm{D}[f], \mathrm{D}[e] & \\ \mathrm{C}[e], \mathrm{D}[f] \to \mathrm{D}, \mathrm{C} & (e\neq f) \\ \mathrm{E}, \mathrm{D} \to \mathrm{D}, \mathrm{E} & \end{array}$$

and

$$\mathrm{A}[e; e', e''], \mathrm{D}[f] \to \varnothing \qquad (f=e' \text{ or } f=e''),$$

we may assume no edge f is introduced by (A) and later deleted by (D).

Suppose e is uncontracted by deduction (C) and not deleted by any later deduction (D). Then e remains in the final graph H, the series–parallel graph associated to Q. $x=l(e)$ is then a variable occurring in $P\leqslant Q$. There can be no applications of (B) in $\mathscr{D}$ to edges with label x. Otherwise the edge introduced by the last one would remain in H, but e is the only edge with label x in H because x appears only once in Q.

Since x also appears in P, there is an edge f with $l(e)=x$ in the initial graph G, the series–parallel graph associated to P. There can be no application of (A) duplicating f. For the duplicate edges f', f'' would have to be deleted by (D), be coalesced by (B), or remain in H, but we have already ruled out each of these possible fates. Of the same possible fates for f itself, the last two are again ruled out. Therefore f is eventually deleted by an instance of (D).

Introduce new variables x_0 and x_1. Change the label of f to x_1 and the label of e to x_0. Since no application of (A) or (B) involves e or f, and the other deductions do not care about labels, we get a proof of

$$P|_{x=x_1} \leqslant Q|_{x=x_0}. \tag{2.5}$$

The proof of Proposition 2.1 shows that (2.5) is equivalent to

$$P|_{x=1} \leqslant Q|_{x=0}.$$

Since this contradicts that $P \leqslant Q$ is short, we conclude that every edge uncontracted by a deduction (C) is later deleted by a deduction (D).

Suppose e is deleted by a deduction (D) and that e was not uncontracted by an earlier deduction (C). An argument symmetric to the preceding one shows that e was in G and again $x = l(e)$ is a variable. As before, no instance of (A) or (B) can involve edges with label y, and the single y appearing in Q corresponds to an edge f in H which could only have been uncontracted by deduction (C). Changing the lables of e and f to new variables y_1 and y_0 we once again get a proof of an identity shorter than $P \leqslant Q$.

We have now shown that every edge uncontracted by (C) is later deleted by (D) and vice versa. Let e be such an edge. Using rules

$$\begin{aligned}
&\mathrm{C, E} \to \mathrm{E, C}\\
&\mathrm{E, D} \to \mathrm{D, E}\\
&\mathrm{C}[e], \mathrm{D}[f] \to \mathrm{D, C} \qquad (e \neq f)\\
&\mathrm{C}[e], \mathrm{C}[f] \to \mathrm{C}[f], \mathrm{C}[e]\\
&\mathrm{D}[e], \mathrm{D}[f] \to \mathrm{D}[f], \mathrm{D}[e],
\end{aligned}$$

we can arrange for each such edge that $\mathrm{D}[e]$ immediately follows $\mathrm{C}[e]$. Thus we obtain a proof satisfying (III) as well as (I) and (II), proving the Theorem. ∎

EXAMPLE 2.3. One form of the *modular law* is the identity

$$(b \vee c) \wedge a \leqslant b \vee (c \wedge (a \vee b)).$$

Proposition 2.1 converts this identity to

$$((b_1 \wedge b_2) \vee c) \wedge a \leqslant b_1 \vee (c \wedge (a \vee b_2)). \tag{2.6}$$

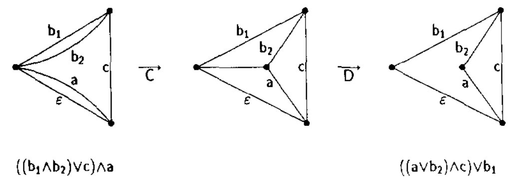

FIG. 5. Proof of an identity equivalent to the modular law.

A proof of (2.6) in normal form is shown in Fig. 5. It consists of a single application of the paired deduction (CD) which partitions vertex v into v_0, v_1. The proof is planar, and after exchanging the symbols a and b_1, self-dual. The reader can amuse him- or herself verifying that (2.6) is short (although the proof would still be valid even if it were not). In fact, replacing any variable in (2.6) by 1 on the left and 0 on the right yields either a form of the distributive law or an identity which fails in the two-element lattice $\{0, 1\}$.

EXAMPLE 2.4. In Example 2.2 we obtained a form (2.2) of the *Arguesian identity* in which every variable appears once on each side. We repeat it here for the reader's convenience:

$$\begin{aligned} c \wedge ([((a_1 \wedge a_2) \vee (a_1' \wedge a_2')) \\ \wedge ((b_1 \wedge b_2) \vee (b_1' \wedge b_2'))] \vee (c_1' \wedge c_2')) \\ \leqslant a_1 \vee ([((a_2 \vee b_1) \wedge (a_2' \vee b_1')) \\ \vee ((b_2 \vee c) \wedge (b_2' \vee c_1'))] \wedge (a_1' \vee c_2')). \end{aligned}$$

Figure 6 shows a proof of (2.2) in normal form. The vertex v is partitioned (in two steps (CD)) into vertices v_0, v_1, v_2; and u similarly. The proof is planar and, after the change of variable names indicated in Example 2.2, self-dual.

Identity (2.2) has a direct geometric interpretation with reference to Fig. 1 (see also Example 2.1). Let a_1 and a_2 both be the point a, and

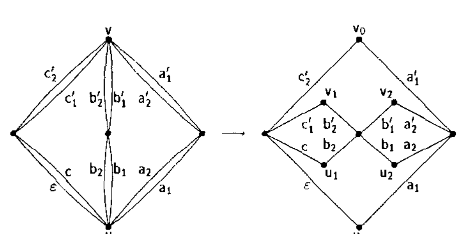

FIG. 6. Proof of the Arguesian identity (2.2).

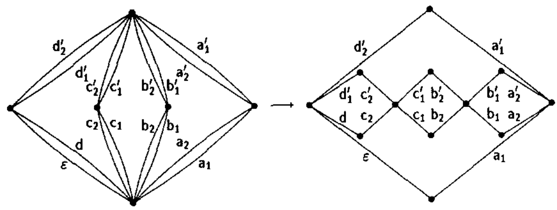

FIG. 7. The first higher Arguesian identity.

similarly for the other letters b, c, a', b', c'. The clause in square brackets $[\cdots]$ on the left side of (2.2) then evaluates to the point p. Joining c' then meeting c gives c (or 0 had the triangles not been centrally perspective). On the right side, the clause in square brackets is the line L through $ab \cap a'b'$ and $bc \cap b'c'$. Meeting the line $a'c'$, then joining a we get a line that passes through c only when the two triangles are axially perspective.

EXAMPLE 2.5. Generalizing Example 2.4 yields *higher Arguesian identities*, the first of which is shown in Fig. 7. It has probably occurred to the reader by now that a complicated lattice polynomial is more easily recognized from a drawing of its series–parallel graph then from an expression for the polynomial itself. Therefore we do not actually write down a higher Arguesian identity, but let the graphs in Fig. 7 stand for its left and right sides. The generalization beyond Figs. 6 and 7 should be apparent. The proofs of the higher Arguesian identities are all analogous to that of the usual Arguesian identity (2.2); all are planar and self-dual.

Each higher Arguesian identity is equivalent by Proposition 2.1 to its special case in which $x_1 = x_2$ for each primed or unprimed letter x among the variables. Using this form and setting $a = b$, $a' = b'$, one easily derives each identity from the next higher one. It is not known whether each higher Arguesian identity is strictly stronger than the ones below it. We suspect that this is indeed the case.

The identity in Fig. 7 has a geometric interpretation. It implies that if aa', bb', cc', and dd' are concurrent lines in projective 3-space, then the four points $ab \cap a'b'$, $bc \cap b'c'$, $cd \cap c'd'$, and $da \cap d'a'$ are coplanar. This geometric configuration is illustrated in Fig. 8.

Word problem conjectures. We can restate Theorem 2.2 in the following way. For each variable x in $P \leqslant Q$ let n_x be a positive integer. $n_x - 1$ represents the number of applications of (A) to edges labelled x. Let G and H be the series–parallel graphs associated to P and Q. Form G^n by replacing each edge labelled x in G with n_x parallel copies. Form H^n by replacing each edge labelled x in H by a path of n_x edges. Then $P \leqslant Q$ has a proof in normal form if and only if for some choice of the n_x, H^n can be obtained from G^n by alternately partitioning vertices and reversing

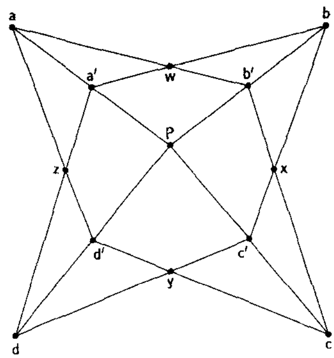

FIG. 8. Geometric interpretation of the first higher Arguesian identity. In the 3-dimensional figure, points w, x, y, and z are coplanar.

series–parallel subgraphs via deduction (E). For specified n_x it is plainly possible to enumerate all graphs obtainable from G^n in this way. Hence if there is some computable bound on the n_x required for a proof of $P \leqslant Q$, Theorem 2.2 would be a solution to the word problem for free linear lattices.

For a given short, valid linear lattice identity $P \leqslant Q$ in which each variable appears exactly once on each side, let $a(P \leqslant Q)$ be the minimum number of applications of (A) (= number of applications of (B) $= \sum_x n_x - 1$) in a normal form proof of $P \leqslant Q$. For each integer $n \geqslant 1$, let $b(n)$ be the maximum of $a(P \leqslant Q)$ over all short, valid linear lattice identities $P \leqslant Q$ in n variables each of which appears exactly once on each side.

Conjecture 2.1. $b(n)$ has a recursive bound.

Conjecture 2.2. $b(n) = 0$.

Note that Conjecture 2.2, if true, would actually make Theorem 2.2 a fairly efficient solution of the free linear lattice word problem. Although we cannot yet prove even Conjecture 2.1, we do have a suggestive result in the direction of Conjecture 2.2.

PROPOSITION 2.2. *Let $b'(n)$ be the maximum of $a(P \leqslant Q)$ taken only over short identities having a normal form proof not using* (E). *Then $b'(n) = 0$.*

Proof. Essentially, the Proposition says that a proof not using (E) need not use (A) or (B) either. Let $\mathscr{D}$ be a normal form proof of $P \leqslant Q$ not using (E) and using a minimum number of applications of (A) and (B). Suppose this number is not zero. Then some $n_x \geqslant 2$ and there are series edges e, f in H^n with $l(e) = l(f) = x$. Let the vertices incident with e and f be u, v and v,

w, respectively. Since there are no applications of (E), G^n is the image of H^n under a gaph homomorphism ϕ. Since e, f are parallel in G^n, $\phi(u) = \phi(w)$, and ϕ factors:

$$H^n \xrightarrow[\theta_{uw}]{} K \xrightarrow[\phi']{} G^n,$$

where θ_{uw} identifies vertices u, w of H^n. Set $n'_x = n_x - 2$, $n'_y = n_y$ $(y \neq x)$. Then $H^{n'} \cong K \backslash \{e, f\}$ and ϕ' induces a graph homomorphism bijective on the edge sets from $K \backslash \{e, f\}$ onto $G^n \backslash \{e, f\} = G^{n'}$. If $n'_x \neq 0$ we get a proof of $P|_{x=1} \leqslant Q|_{x=0'}$ contradicting shortness. Otherwise we get a proof $P \leqslant Q$ using two fewer applications each of (A) and (B) than $\mathscr{D}$, contradicting that $\mathscr{D}$ had a minimum number. ∎

Example 2.6 gives a short linear lattice identity with no proof not using (E). Thus one cannot hope to get Conjecture 2.2 as a direct consequence of Proposition 2.2 by eliminating deduction (E). However, the identity in Example 2.6 is a consequence of the modular law, which does have a proof not using (E) (Example 2.3).

Example 2.7 gives a linear lattice identity whose dual has no proof if Conjecture 2.2 holds. Conjecture 2.2 would thus imply that the variety generated by linear lattices is not self-dual. In particular, it could not then be equal to the variety of Arguesian lattices. Whether these varieties are equal is a famous question of Jónsson [36, 25].

EXAMPLE 2.6. The identity

$$[(a \wedge b \wedge c) \vee d] \wedge [(a' \wedge b' \wedge c') \vee d'] \leqslant [(a \vee b') \wedge (a' \vee b)] \vee [(c' \vee d) \wedge (c \vee d')] \tag{2.7}$$

has the normal form proof shown in Fig. 9. (2.7) is easily verified to be short. In fact, replacing any variable by 1 on the left side and 0 on the right

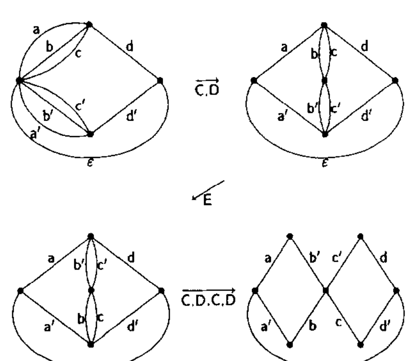

FIG. 9. Proof of a short identity requiring deduction (E).

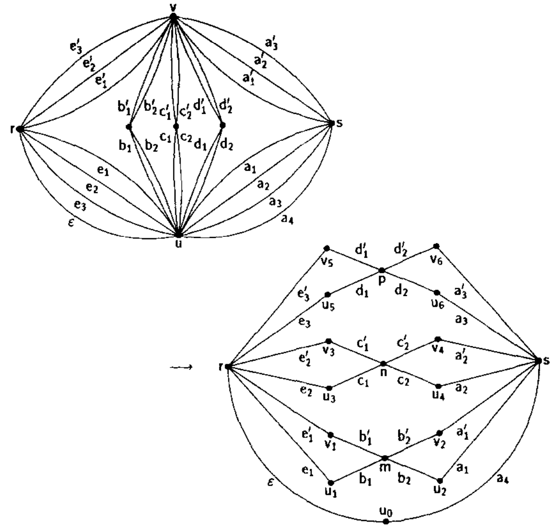

FIG. 10. An identity whose natural proof is not planar.

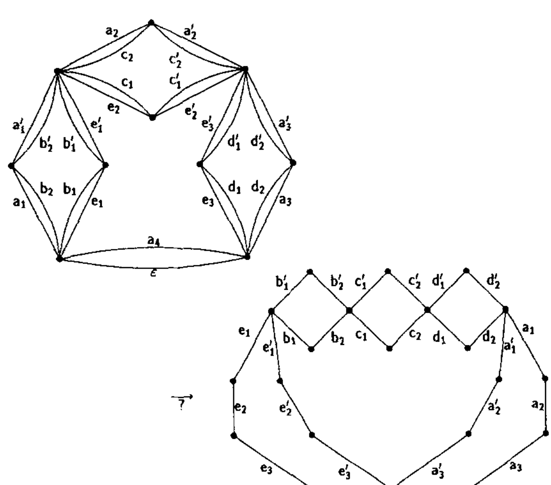

FIG. 11. The dual identity to that of Fig. 10. Any proof would require deductions (A) and (B).

side yields an identity which fails in the two-element lattice $\{0, 1\}$. It is also easy to see that no series–parallel graph equivalent to (2.4) of Fig. 9 can be obtained by partitioning vertices of a series–parallel graph equivalent to (2.1). Proposition 2.2 then shows that every normal form proof of (2.7) must use (E).

EXAMPLE 2.7. The identity shown in Fig. 10 (the actual lattice polynomials are too complicated to usefully write down) is proved by partitioning vertices u and v as indicated. This proof is not planar. The dual identity is shown in Fig. 11. It is not hard, though it is somewhat tedious, to check that the identity in Fig. 11 cannot have a proof using only vertex partitioning and deduction (E). Therefore, if it has a proof at all, such a proof must use deductions (A) and (B). Since it can also be checked that no shortening of the identity in Fig. 11 is valid in all linear lattices, the identity itself cannot be valid if Conjecture 2.2 holds.

The identity of Fig. 11 can be shown to hold in all congruence lattices of Mal'cev algebras. If it fails for linear lattices in general, we would therefore have a new proof that not every linear lattice is in the variety generated by congruence lattices of Mal'cev algebras [10].

3.0. CONCLUDING REMARKS

The four most fundamental open questions about the class of linear lattices are (1) is it a variety, (2) is it self-dual, (3) does it generate the variety of Arguesian lattices, and (4) does it have a solvable free lattice word problem? The diagrammatic proof theory presented here has clear relevance for attacks on the last three questions, as we have sought to make clear. A possible proof-theoretic approach to the variety question would be to normalize Horn sentence proofs so that substitutions (F) occur only in series–parallel graphs, thus "factoring" Horn sentence proofs into identity proofs and general lattice-theoretic substitutions. It is possible to so factor a proof of the Arguesian implication (Example 1.1), but we do not yet see any general method to accomplish such a factoring. Regarding the word problem question, we note that the results of [21, 22, 27] prove that the general word problem for linear lattices is unsolvable.

In separate publications, we plan to extend and apply the methods introduced in this paper in several directions, two of which we briefly mention. In the author's doctoral thesis, it is shown that a reformulation of linear lattice proof theory in terms of graphic matroids yields a proof theory for lattices of Mal'cev algebra "pseudocongruences" (equivalence relations admissible through the middle variable of the ternary Mal'cev term). Using other classes of matroids, one obtains proof theories for other classes of modular lattices, including a new and useful calculus for modular lattice computations. Combinatorial methods for separating these proof theories may allow separation of, for instance, the higher Arguesian identities (Example 2.5) without requiring the construction of explicit counterexamples.

Another direction of research is the coordinatization of linear lattices, following that of [33] for modular lattices, subsequently improved by [7] for Arguesian lattices. In linear lattices, we can easily and elegantly obtain the Abelian group part of this coordinatization without reference to the multiplicative structure and relative to a more general (and more obviously projective) configuration than the "n-frames" used formerly.

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